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Lyapunov and Wirtinger Inequalities

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Abstract—In this paper, we prove the Lyapunov inequality for the second-order linear differential equation

$$(r(t)\phi(y'(t)))' + p(t)\phi(y(t)) = 0,$$

where

(i) $\phi(s) = |s|^{\alpha-2}s$, $\alpha > 1$ is a fixed real number,
(ii) $r(t)$ and $p(t)$ are integrable on $[a, b]$ with $r(t) > 0$ on $[a, b]$.

On the other hand, a generalized Wirtinger inequality is also given.

Keywords—Lyapunov's inequality, Hölder's inequality, Wirtinger's inequality.

1. INTRODUCTION

The purpose of this note is to generalize the Lyapunov inequality [1–5] for the second-order linear differential equation

$$y''(x) + p(x)y(x) = 0$$

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to the half-linear differential equation

$$(r(t)\phi(y'(t)))' + p(t)\phi(y(t)) = 0, \tag{1}$$

where

(i) $\phi(s) = |s|^{\alpha-2}s$, $\alpha > 1$ is a fixed real number,
(ii) $r(t)$ and $p(t)$ are integrable on $[a,b]$ with $r(t) > 0$ on $[a,b]$.

For other related results, we refer to [6,7].

In Section 3, we also extend the well-known Wirtinger inequality to the more general case.

2. LYAPUNOV'S INEQUALITY

In order to prove our main result, we need the following lemma.

LEMMA 1. *Let a and b be successive zeros of a nontrivial solution $u(t)$ of the half-linear differential equation (1). Then, there exists $c \in (a,b)$ such that*

$$\left[\int_a^c \left(\frac{1}{r(s)}\right)^{\beta/\alpha} ds\right]^{1/\beta} \left[\int_a^c p^+(s)\,ds\right]^{1/\alpha} > 1 \tag{2}$$

and

$$\left[\int_c^b \left(\frac{1}{r(s)}\right)^{\beta/\alpha} ds\right]^{1/\beta} \left[\int_c^b p^+(s)\,ds\right]^{1/\alpha} > 1, \tag{3}$$

where $1/\alpha + 1/\beta = 1$.

PROOF. It follows from $u(a) = u(b) = 0$ and Rolle's theorem that there exists $c \in (a,b)$ such that $u'(c) = 0$. Clearly, $|u(c)| = \max_{t\in[a,b]} |u(t)| \neq 0$. Using the Hölder inequality, we have

$$\begin{aligned}
|u(c)| &= \left|\int_a^c u'(s)\,ds\right| \\
&\le \int_a^c |u'(s)|\,ds \\
&= \int_a^c \left(\frac{1}{r(s)}\right)^{1/\alpha} r(s)^{1/\alpha}|u'(s)|\,ds \\
&< \left(\int_a^c \left(\frac{1}{r(s)}\right)^{\beta/\alpha} ds\right)^{1/\beta} \left(\int_a^c r(s)|u'(s)|^\alpha\,dt\right)^{1/\alpha} \\
&= \left(\int_a^c \left(\frac{1}{r(s)}\right)^{\beta/\alpha} ds\right)^{1/\beta} \left(\int_a^c p(s)|u(s)|^\alpha\,ds\right)^{1/\alpha} \\
&\le |u(c)| \left(\int_a^c \left(\frac{1}{r(s)}\right)^{\beta/\alpha} ds\right)^{1/\beta} \left(\int_a^c p^+(s)\,ds\right)^{1/\alpha}.
\end{aligned}$$

Hence,

$$\left(\int_a^c \left(\frac{1}{r(s)}\right)^{\beta/\alpha} ds\right)^{1/\beta} \left(\int_a^c p^+(s)\,ds\right)^{1/\alpha} > 1.$$

Similarly,

$$\begin{aligned}
|u(c)| = |-u(c)| &= \left| \int_c^b u'(s)\,ds \right| \\
&\le \int_c^b |u'(s)|\,ds \\
&= \int_c^b \left(\frac{1}{r(s)}\right)^{1/\alpha} r(s)^{1/\alpha} |u'(s)|\,ds \\
&< \left(\int_c^b \left(\frac{1}{r(s)}\right)^{\beta/\alpha} ds\right)^{1/\beta} \left(\int_c^b r(s)|u'(s)|^\alpha\,dt\right)^{1/\alpha} \\
&= \left(\int_c^b \left(\frac{1}{r(s)}\right)^{\beta/\alpha} ds\right)^{1/\beta} \left(\int_c^b p(s)|u(s)|^\alpha\,ds\right)^{1/\alpha} \\
&\le |u(c)| \left(\int_c^b \left(\frac{1}{r(s)}\right)^{\beta/\alpha} ds\right)^{1/\beta} \left(\int_c^b p^+(s)\,ds\right)^{1/\alpha}.
\end{aligned}$$

Hence,

$$\left(\int_c^b \left(\frac{1}{r(s)}\right)^{\beta/\alpha} ds\right)^{1/\beta} \left(\int_c^b p^+(s)\,ds\right)^{1/\alpha} > 1.$$

Thus, we complete our proof.

We now can state and prove our main two results as follows.

THEOREM 2. *Let a and b be successive zeros of a nontrivial solution $u(t)$ of the differential equation*

$$\left(|y'(t)|^{\alpha-2}y'(t)\right)' + g(t)|y'(t)|^{\alpha-2}y'(t) + f(t)|y(t)|^{\alpha-2}y(t) = 0, \tag{4}$$

where

(a) *$\alpha \ge 2$ is a fixed constant,*
(b) *$f(t)$ and $g(t)$ are integrable on $[a,b]$.*

Then,

$$(b-a)^{\alpha/\beta} \int_a^b f^+(s)\,ds - 4\exp\left(-\int_a^b |g(s)|\,ds\right) > 0 \tag{R$_1$}$$

and

$$(b-a)^{\alpha/\beta} \int_a^b f^+(s)\,ds + 4\int_a^b |g(s)|\,ds > 4, \tag{R$_2$}$$

where $1/\alpha + 1/\beta = 1$.

PROOF. Let

$$r(t) = \exp\left(\int_a^t g(s)\,ds\right) \qquad \text{and} \qquad p(t) = r(t)f(t)$$

or

$$r(t) = \exp\left(-\int_t^b g(s)\,ds\right) \qquad \text{and} \qquad p(t) = r(t)f(t).$$

Then, equation (4) reduces to equation (1). By Lemma 1, there exists $c \in (a,b)$ such that

$$\left[\int_a^c \left(\exp\left(-\int_a^t g(s)\,ds\right)\right)^{\beta/\alpha} dt\right]^{1/\beta} \left[\int_a^c f^+(t)\exp\left(\int_a^t g(s)\,ds\right) dt\right]^{1/\alpha} > 1$$

and

$$\left[\int_c^b \left(\exp\left(\int_t^b g(s)\,ds\right)\right)^{\beta/\alpha} dt\right]^{1/\beta} \left[\int_c^b f^+(t)\exp\left(-\int_t^b g(s)\,ds\right) dt\right]^{1/\alpha} > 1.$$

Thus,

$$(c-a)^{\alpha/\beta}\left(\int_a^c f^+(t)\,dt\right)\exp\left(2\int_a^c |g(s)|\,ds\right) > 1 \tag{5}$$

and

$$(b-c)^{\alpha/\beta}\left(\int_c^b f^+(t)\,dt\right)\exp\left(2\int_c^b |g(s)|\,ds\right) > 1. \tag{6}$$

Let

$$\begin{aligned}
A_1 &:= \sqrt{(c-a)^{\alpha/\beta}\left(\int_a^c f^+(t)\,dt\right)},\\
B_1 &:= \sqrt{(b-c)^{\alpha/\beta}\left(\int_c^b f^+(t)\,dt\right)},\\
A_0 &:= 2\int_a^c |g(s)|\,ds,\\
B_0 &:= 2\int_c^b |g(s)|\,ds,\\
K &:= \left(\frac{A_1^2}{B_1^2}\right)^{1/\alpha}.
\end{aligned}$$

These and (5),(6) imply

$$A_1^2 > e^{-A_0} > 1 - A_0 \tag{7}$$

and

$$B_1^2 > e^{-B_0} > 1 - B_0. \tag{8}$$

Clearly, it follows from $\alpha/\beta = \alpha - 1 \geq 1$ that

$$\begin{aligned}
\int_a^b f^+(t)\,dt &= \int_a^c f^+(t)\,dt + \int_c^b f^+(t)\,dt\\
&= \frac{A_1^2}{(c-a)^{\alpha/\beta}} + \frac{B_1^2}{(b-c)^{\alpha/\beta}}\\
&\geq \frac{A_1^2}{((Kb+a)/(K+1)-a)^{\alpha/\beta}} + \frac{B_1^2}{(b-(Kb+a)/(K+1))^{\alpha/\beta}}\\
&= \frac{(K+1)^{\alpha/\beta}A_1^2}{K^{\alpha/\beta}(b-a)^{\alpha/\beta}} + \frac{(K+1)^{\alpha/\beta}B_1^2}{(b-a)^{\alpha/\beta}}\\
&= \left(\frac{K+1}{K}\right)^{\alpha/\beta}\left[\frac{A_1^2 + K^{\alpha/\beta}B_1^2}{(b-a)^{\alpha/\beta}}\right]\\
&\geq \left[1 + \left(\frac{1}{K}\right)^{\alpha/\beta}\right]\left[\frac{A_1^2 + K^{\alpha/\beta}B_1^2}{(b-a)^{\alpha/\beta}}\right]\\
&= \frac{A_1^2 + K^{\alpha/\beta}B_1^2 + (1/K)^{\alpha/\beta}A_1^2 + B_1^2}{(b-a)^{\alpha/\beta}}\\
&= \frac{A_1^2 + A_1^{2/\beta}B_1^{2/\alpha} + B_1^{2/\beta}A_1^{2/\alpha} + B_1^2}{(b-a)^{\alpha/\beta}}.
\end{aligned} \tag{9}$$

It follows from (7)–(9) and the inequality

$$e^{-x} + e^{-y} \geq 2e^{-(x+y)/2}, \qquad \text{for all } x, y \in \mathbb{R}, \tag{10}$$

that

$$\begin{aligned}
(b-a)^{\alpha/\beta} \int_a^b f^+(s)\,ds &- 4\exp\left(-\left(\int_a^c |g(s)|\,ds + \int_c^b |g(s)|\,ds\right)\right) \\
&\geq A_1^2 + A_1^{2/\beta} B_1^{2/\alpha} + B_1^{2/\beta} A_1^{2/\alpha} + B_1^2 - 4e^{-(A_0+B_0)/2} \\
&> e^{-A_0} + e^{-(A_0/\beta + B_0/\alpha)} + e^{-(B_0/\beta + A_0/\alpha)} + e^{-B_0} - 4e^{-(A_0+B_0)/2} \\
&\geq e^{-A_0} + e^{-B_0} - 2e^{-(A_0+B_0)/2} \\
&= \left(e^{-A_0/2} - e^{-B_0/2}\right)^2 \geq 0.
\end{aligned}$$

Hence, (R_1) holds. It follows from

$$e^{-x} > 1 - x, \qquad \text{for all } x > 0,$$

that (R_2) holds. Thus, the proof of Theorem 1 is complete.

REMARK 3. If $\alpha = 2$ and $g(t) = 0$, then (R_2) reduces to the classical Lyapunov inequality.

THEOREM 4. *Let a and b be successive zeros of a nontrivial solution $u(t)$ of the differential equation*

$$\left(|y'(t)|^{\alpha-2} y'(t)\right)' + g(t)|y'(t)|^{\alpha-2} y'(t) + f(t)|y(t)|^{\alpha-2} y(t) = 0,$$

where

(c) $1 < \alpha \leq 2$ *is a fixed constant,*
(d) $f(t)$ *and* $g(t)$ *are integrable on* $[a, b]$.

Then,

$$(b-a)^{\alpha/\beta} \int_a^b f^+(s)\,ds - 2^\alpha \exp\left(-\int_a^b |g(s)|\,ds\right) > 0 \tag{R_1'}$$

and

$$(b-a)^{\alpha/\beta} \int_a^b f^+(s)\,ds + 2^\alpha \int_a^b |g(s)|\,ds > 2^\alpha, \tag{R_2'}$$

where $1/\alpha + 1/\beta = 1$.

PROOF. It follows from $1 < \alpha < 2$ that $0 < \alpha/\beta = \alpha - 1 < 1$. Therefore, the inequality

$$(1+x)^{\alpha/\beta} \geq 2^{\alpha/\beta - 1}\left(1 + x^{\alpha/\beta}\right), \qquad \text{for all } x \geq 0,$$

holds. Then,

$$\begin{aligned}
\int_a^b f^+(t)\,dt &\geq \left(\frac{K+1}{K}\right)^{\alpha/\beta} \left[\frac{A_1^2 + K^{\alpha/\beta} B_1^2}{(b-a)^{\alpha/\beta}}\right] \\
&\geq 2^{(\alpha/\beta)-1} \left[1 + \left(\frac{1}{K}\right)^{\alpha/\beta}\right] \left[\frac{A_1^2 + K^{\alpha/\beta} B_1^2}{(b-a)^{\alpha/\beta}}\right] \\
&= 2^{(\alpha/\beta)-1} \left[\frac{A_1^2 + K^{\alpha/\beta} B_1^2 + (1/K)^{\alpha/\beta} A_1^2 + B_1^2}{(b-a)^{\alpha/\beta}}\right] \\
&= 2^{(\alpha/\beta)-1} \left[\frac{A_1^2 + A_1^{2/\beta} B_1^{2/\alpha} + B_1^{2/\beta} A_1^{2/\alpha} + B_1^2}{(b-a)^{\alpha/\beta}}\right],
\end{aligned} \tag{11}$$

where K, A_1, B_1 are defined as in the proof of Theorem 1. Thus,

$$2^{1-\alpha/\beta}(b-a)^{\alpha/\beta} \int_a^b f^+(s)\,ds - 4\exp\left(-\left(\int_a^c |g(s)|\,ds + \int_c^b |g(s)|\,ds\right)\right) > 0.$$

Hence, (R_1') and (R_2') hold. This completes the proof of Theorem 2.

3. WIRTINGER'S INEQUALITY

Let L be the differential operator defined by

$$Lv = \left(p|v'|^{\alpha-2}v'\right)' + q|v|^{\alpha-2}v, \tag{12}$$

where $p, q \in C\ (I = (a,b), \mathbb{R})$. Here $-\infty < a < b < \infty$. The domain D of L is defined to be the set of all real valued functions v in I such that all derivatives involved in (12) exist and are continuous at each point in I. We shall consider solutions $v \in D$ of the differential inequality

$$-Lv \geq \lambda_0 rv \tag{13}$$

in I, where λ_0 is a real number and r is a positive continuous function in I.

Let AC(I) denote the set of all real valued functions which are absolutely continuous on every closed subinterval of I. For a positive solution $v \in D$ of (13), we consider functions $u \in \mathrm{AC}(I)$ such that the limits below exist and are finite

$$\begin{aligned} S_1(u,v) &= \lim_{x\to a^+} \frac{p(x)u^2(x)|v'(x)|^{\alpha-2}}{v(x)}, \\ S_2(u,v) &= \lim_{x\to b^-} \frac{p(x)u^2(x)|v'(x)|^{\alpha-2}}{v(x)}. \end{aligned} \tag{14}$$

THEOREM 5. *Let $v \in D$ be a positive solution of (13) in I for some real number λ_0. If $u \in \mathrm{AC}(I)$, such that the limits in (14) exist and are finite, then the following inequality holds if the integral exists:*

$$\int_I (q+\lambda_0)u^2|v|^{\alpha-2}\,dx \leq \int_I p(u')^2|v'|^{\alpha-2}\,dx + S_1(u,v) - S_2(u,v), \tag{15}$$

where equality holds if and only if $u(x)$ is a constant multiple of $v(x)$ on I.

PROOF. Clearly, we have the differential identity

$$pv^2|v'|^{\alpha-2}\left[\left(\frac{u}{v}\right)'\right]^2 + \left[\frac{pu^2|v'|^{\alpha-2}v'}{v}\right]' = p|v'|^{\alpha-2}(u')^2 + \frac{u^2}{v}Lv - qu^2|v|^{\alpha-2}.$$

Integrating the above identity over a subinterval (y,z) of I with $a < y < z < b$,

$$\int_y^z \left[p(u')^2|v'|^{\alpha-2} - (q+\lambda_0 r)u^2|v|^{\alpha-2}\right]\,dx \geq \left[\frac{pu^2|v'|^{\alpha-2}v'}{v}\right]_y^z,$$

where equality holds if and only if $(u/v)' = 0$ a.e. in I, that is, $u(x) = (\text{const.})v(x)$ on I. Taking limits as $y \to a^+$ and $z \to b^-$, we obtain conclusion (15).

As a specialization of Theorem 5, we consider the case that $I = (a,b)$ is bounded, the continuity of p, q, and r extends to $[a,b]$, and v is a positive eigenfunction corresponding to the smallest eigenvalue λ_0 of the boundary value problem

$$\begin{cases} Lv + \lambda_0 rv = 0, & \text{in } I, \\ \begin{cases} c_1 v(a) - p(a)|v'(a)|^{\alpha-2} = 0, \\ c_2 v(b) + p(b)|v'(b)|^{\alpha-2} = 0, \end{cases} \end{cases} \tag{16}$$

where c_1 and c_2 are constants.

THEOREM 6. *Let $v \in D$ be a positive eigenfunction in a bounded interval I corresponding to the smallest eigenvalue λ_0 of (16). Then, every absolutely continuous function u on $[a,b]$ such that $u' \in L^2(a,b)$ satisfies the following Wirtinger-type inequality if the integral exists:*

$$\int_a^b (q+\lambda_0)u^2|v|^{\alpha-2}\,dx \le \int_a^b p(u')^2|v'|^{\alpha-2}\,dx + c_1u^2(a) + c_2u^2(b), \tag{17}$$

where equality holds if and only if $u(x)$ is a constant multiple of $v(x)$ on $[a,b]$.

REMARK 7. Let $\alpha = 2$. Then, Theorems 5 and 6 reduce to Theorems 1 and 2 in [8], respectively.

REMARK 8. Let $p(x) = 1$, $\lambda_0 = 0$, $\alpha = 2$, $c_2 \le 0$, $c_1 = \infty$, $u(a) = 0$, and $a = 0$ in Theorem 6. Then, (17) reduces to Theorem 1.1 of [9] as follows:

$$\int_a^b (u'(x))^2\,dx \ge \int_a^b q(x)u^2(x)\,dx,$$

where equality holds if and only if $u(x) = (\text{const.})v(x)$.

Let $p(x) = 1$, $q(x) = 0$, $r(x) = 1$, $\alpha = 2$, $\lambda_0 = \pi^2/4(b-a)^2$, $c_1 = \infty$, $c_2 = 0$, $u(x) = w(x) - w(a)$, and $v(x) = \sin[\pi(x-a)/2(b-a)]$ in Theorem 6. Then, (17) reduces to the following corollary.

COROLLARY 9. *(See [8].) Every real valued function $w \in C^1[a,b]$ satisfies the inequality*

$$\int_a^b [w(x) - w(a)]^2\,dx \le 4\left(\frac{b-a}{\pi}\right)^2 \int_a^b [w'(x)]^2\,dx,$$

where equality holds if and only if

$$w(x) = w(a) + K\sin\left(\frac{\pi(x-a)}{2(b-a)}\right)$$

identically on $[a,b]$ for some constant K.

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